TD : Nombres réels

Nombres algébriques

1. a) On veut que : P n'admet pas de racine rationnelle.

Par l'absurde, supposons que P admette $\frac{p}{q}$ comme une racine avec $p \in \mathbb{Z}$, $q \in \mathbb{N}^*$.
Alors $(qX-p) \mid P$. D'où, contradiction.

$$|P(x)|\, q^d \in \mathbb{N} \setminus \{0\}.$$

Donc, $|P(x)|\, q^d \geq 1$.

$$|P(x)| \geq \frac{1}{q^d}.$$

b) Sur $[\alpha-1, \alpha+1]$.

$$|P(x)-P(\alpha)| \leq \underbrace{\|P'\|_{\infty, [\alpha-1,\alpha+1]}}_{\neq 0} \times |x-\alpha|$$

Pour tout $x = \frac{p}{q} \in [\alpha-1, \alpha+1]$.

Soit $C = \dfrac{1}{\|P'\|_{\infty, [\alpha-1,\alpha+1]}}$

$$|x-\alpha| \geq |P(x)| \times C \geq \frac{C}{q^d}.$$

2.

$$\left| x_\varepsilon - \frac{\varepsilon_1 \cdot 2^{n!-1!} + \cdots + \varepsilon_n}{2^{n!}} \right| \leq 2 \times \left(\frac{1}{2^{n!}}\right)^{n+1} \quad (*)$$

$$\left| x_\varepsilon - \frac{p_n}{2^{n!}} \right| \geq \frac{C}{(2^{n!})^d} \text{ est impossible pour } n \text{ grand.}$$

$$\sum_{i=n+1}^{\infty} \frac{1}{2^{i!}} \leq \frac{1}{2^{\varepsilon_i!}}\left[1+\frac{1}{2}+\cdots\right] = 2 \times \left(\frac{1}{2^{n!}}\right)^{n+1}$$

<u>Mq:</u> x_ε est irrationnel.

Par ABS, $x_\varepsilon = \frac{p}{q}$

$$\left| \frac{p \cdot 2^{n!} - p_n q}{2^{n!} \cdot q} \right| \leq 2 \times \frac{1}{(2^{n!})^{n+1}}$$

dès que : $2^{n!} > 2q$, $\left| \frac{p}{q} - \frac{p_n}{q_n} \right| = 0$

Or p_n est impair

$$p \cdot 2^{n!} = \underbrace{p_n}_{\text{impair}} \cdot q \quad \text{donc } 2^{n!} \mid q$$

$$\Rightarrow 2^{n!} \leq q < 2 \cdot 2^{n!}$$

$\lightning$

Fractions continues

1)

$\underline{\text{Mq}}$: $[a_0, \dots, a_n] = [a_0, \dots, a_{n-1} + \frac{1}{a_n}]$

Posons $F(i,j) = [a_i, \dots, a_j]$, $G(i,j) = [a_i, \dots, a_j + \frac{1}{a_{j+1}}]$

$$F(0,n) = G(0,n-1)$$

$$\iff a_0 + \frac{1}{F(1,n)} = a_0 + \frac{1}{G(1,n-1)}$$

$$\iff F(1,n) = G(1,n-1)$$

$$\iff F(n-1,n) = G(n-1,n-1)$$

$$\iff [a_{n-1}, a_n] = [a_{n-1} + \frac{1}{a_n}]$$

Hyp. de récurrence, sur la longueur,

$F(1,n) = \underbrace{\tilde{F}(0,n-1)}_{a_1, \dots, a_n - 1}$

$G(1,n-1) = \underbrace{\tilde{G}(0,n-2)}_{a_1, \dots, a_{n-1}}$

2)

$\underline{n=0}$:

$$\frac{p_0}{q_0} = a_0 = [a_0]$$

$\underline{n=1}$:

$$[a_0, a_1] = a_0 + \frac{1}{a_1} = \frac{a_0 a_1 + 1}{a_1} = \frac{p_1}{q_1}$$

Supposons que pour n, $[a_0, \dots, a_n] = \frac{p_n}{q_n}$

Pour $n+1$, $[a_0, \dots, a_n, a_{n+1}]$

$$= [a_0, \dots, a_n + \frac{1}{a_{n+1}}]$$

$$= \frac{p_n'}{q_n'} = \frac{(a_n + \frac{1}{a_{n+1}}) p_{n-1} + p_{n-2}}{(a_n + \frac{1}{a_{n+1}}) q_{n-1} + q_{n-2}}$$

$p_{n-1}, p_{n-2}, \dots$ ne dépendent que de $a_0, \dots, a_{n-1}$

$$= \frac{(a_{n+1} a_n + 1) p_{n-1} + a_{n+1} p_{n-2}}{(a_{n+1} a_n + 1) q_{n-1} + q_{n-2} a_{n+1}}$$

$$= \frac{a_{n+1} (a_n p_{n-1} + p_{n-2}) + p_{n-1}}{a_{n+1} (a_n q_{n-1} + q_{n-2}) + q_{n-1}}$$

Déf $\left\{ = \frac{a_{n+1} p_n + p_{n-1}}{a_{n+1} q_n + q_{n-1}} = \frac{p_{n+1}}{q_{n+1}} \right.$

3)

$$p_0 q_1 - p_1 q_0 = a_0 a_1 - (a_0 a_1 + 1) = -1$$

$$p_{n+1} q_{n+2} - p_{n+2} q_{n+1} = p_{n+1}(a_{n+2} q_{n+1} + q_n) - (a_{n+2} p_{n+1} + p_n) q_{n+1}$$
$$= p_{n+1} q_n - p_n q_{n+1}$$
$$= (-1)^{n+2}$$

4) On observe $\forall n \in \mathbb{N}$, $\theta_n \in]1, +\infty[\setminus \mathbb{Q}$
$\theta_n > a_n \geq 1$.

a) Pour $n \in \mathbb{N}$,
$$[a_0, \dots, a_{n+1}, \theta_{n+2}]$$
$$= [a_0, \dots, a_n, \underbrace{a_{n+1} + \frac{1}{\theta_{n+2}}}_{\theta_{n+1}}] \quad \text{relation (1)}$$

$$= [a_0, \dots, a_n, \theta_{n+1}] \text{ cste } = [a_0, \theta_1] = \theta.$$

b) On fixe $n \in \mathbb{N}$.
On note $\varphi_n : x \mapsto [a_0, \dots, a_n, x]$, $x > 0$
$\varphi_0(x) = a_0 + \frac{1}{x}$ décroît

$x \mapsto \frac{ax+b}{cx+d}$.

On va mq : φ_n est une fonction homographique
$\begin{cases} \downarrow \text{ si } n \text{ pair} \\ \uparrow \text{ si } n \text{ impair} \end{cases}$

<u>n pair</u> : $\varphi_n(x) = \frac{\alpha x + \beta}{\gamma x + \delta}$ $\alpha, \beta, \gamma, \delta > 0$

HR : $\varphi_n \downarrow$

$$\varphi_{n+1}(x) = [a_0, \dots, a_{n+1}, x]$$

$$= [a_0, \dots, a_n, a_{n+1} + \frac{1}{x}]$$

$$= \varphi_n\left(a_n + \frac{1}{x}\right)$$

$$= \frac{\alpha\left(a_n + \frac{1}{x}\right) + \beta}{\gamma\left(a_n + \frac{1}{x}\right) + \delta}$$ homographique.

n pair,

$$\theta = [a_0, \dots, a_n, \theta_{n+1}] < \underbrace{[a_0, \dots, a_{n+1}]}_{r_{n+1}}$$

$$= \left[a_0, \dots, a_n + \frac{1}{\theta_{n+1}}\right]$$

$$> \underbrace{[a_0, \dots, a_n]}_{r_n}$$

c) On écrit $\theta = [a_0, \dots, a_n, \theta_{n+1}]$

Alors, $$|\theta - r_n| = \left| \frac{\theta_{n+1} p_n + p_{n-1}}{\theta_{n+1} q_n + q_{n-1}} - \frac{p_n}{q_n} \right|$$

$$= \left| \frac{p_{n-1} q_n - p_n q_{n-1}}{q_n(\theta_{n+1} q_n + q_{n-1})} \right|$$

$$= \frac{1}{q_n(\theta_{n+1} q_n + q_{n-1})}$$

$$\leq \frac{1}{q_n([\theta_{n+1}] q_n + q_{n-1})}$$

$$= \frac{1}{q_n(\theta_{n+1} q_n + q_{n-1})} = \frac{1}{q_n q_{n+1}}$$

$$\Rightarrow |\theta - r_n| < \frac{1}{q_n q_{n+1}} < \frac{1}{q_n^2}$$

Application :

Soit $(x, \varepsilon) \in \mathbb{R} \times \mathbb{R}_+$

On veut mq $\exists y \in \theta\mathbb{N} - \mathbb{N}$ t.q. $|y - x| < \varepsilon$

$\lim_{n \to +\infty} q_n = +\infty$ $\qquad 0 < |\theta q_n - p_n| < \frac{1}{q_n} < \varepsilon$ APCR

Exo :

Soit $\begin{cases} \lambda > 0 \\ \lambda \neq 1 \end{cases}$ α un nombre algébrique réel non rationnel.

Étudier $\dfrac{\lambda^n}{|\cos(n\pi\alpha)|}$

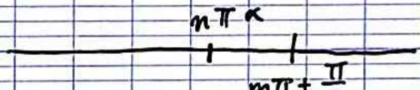
Cas trivial : Si $\lambda > 1$.

$$\frac{\lambda^n}{|\cos(n\pi\alpha)|} \geq \lambda^n \longrightarrow +\infty.$$

Si $\lambda < 1$, par ex $\alpha > 0$.

Soit $n \in \mathbb{N}^*$, $n\alpha \gg \pi$.

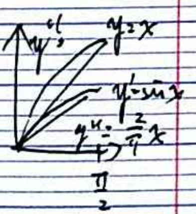
On choisit $m \in \mathbb{N}^*$ t.q. $\underbrace{\left|\pi n\alpha - \left(m\pi + \frac{\pi}{2}\right)\right|}_{\varepsilon_n}$ soit minimal.



$$|\cos \pi n\alpha| = \left|\cos\left(m\pi + \frac{\pi}{2} + \varepsilon_n\right)\right| = |\sin \varepsilon_n|$$

$$\frac{2\varepsilon_n}{\pi} = |2n\alpha - (2m+1)| = 2n\left|\alpha - \frac{2m+1}{2n}\right| \geq \frac{C}{(2n)^d} \times 2n$$

α : algébrique au degré d. $P(X)$



$$|\sin \varepsilon_n| \geq \frac{2}{\pi}\varepsilon_n \geq \frac{C'}{(2n)^{d-1}}$$

$$\frac{\lambda^n}{|\cos(\pi n\alpha)|} \leq \frac{\lambda^n}{\left(\frac{C'}{(2n)^{d-1}}\right)} \leq \left(\frac{1}{C'}\right) \underbrace{\lambda^n (2n)^{d-1}}_{\longrightarrow 0}$$

$$\frac{1}{\sqrt{5}}\left(\lambda+\frac{1}{\lambda}\right) \leq 1$$

$$\Leftrightarrow \lambda^2-\sqrt{5}\lambda+1 \leq 0$$

$$\Leftrightarrow (\lambda-\varphi)\left(\lambda-\frac{1}{\varphi}\right) \leq 0$$

$$\Leftrightarrow \left(\lambda-\left(\frac{1+\sqrt{5}}{2}\right)\right)\left(\lambda-\left(\frac{\sqrt{5}-1}{2}\right)\right) \leq 0$$

$$\frac{\sqrt{5}-1}{2} < \lambda < \frac{1}{2}(\sqrt{5}+1)$$

De même, $\mu \in \left]\frac{1}{\varphi}, \varphi\right[$

$$q_{n+2} = a_{n+2}q_{n+1}+q_n$$

$$\Rightarrow \frac{q_{n+2}}{q_{n+1}} = a_{n+2}+\frac{q_n}{q_{n+1}}$$

$$\Leftrightarrow \mu = a_{n+2}+\frac{1}{\lambda}$$

$$\Leftrightarrow a_{n+2} = \mu-\frac{1}{\lambda} < \varphi-\frac{1}{\varphi}$$

$$\Rightarrow a_{n+2} \in]-1,1[\quad \lightning$$

On choisit $(a_n)=(1)_n$

$$P_n = F_{n+1},\quad q_n = F_n,\quad F_n = \frac{\varphi^{n+1}-\left(-\frac{1}{\varphi}\right)^{n+1}}{\sqrt{5}} \sim \frac{\varphi^{n+1}}{\sqrt{5}}$$

$$\left|\varphi-\frac{p_n}{q_n}\right| = \left|(-1)^n\frac{\left(\frac{1}{\varphi}\right)^n\left(1+\frac{1}{\varphi^2}\right)}{\varphi^{n+1}-\left(-\frac{1}{\varphi}\right)^{n+1}}\right| = \frac{\sqrt{5}}{\varphi^{2n+2}-(-1)^{n+1}} \sim \frac{\sqrt{5}}{\varphi^{2n+2}}$$

$$q_n^2\left|\varphi-\frac{p_n}{q_n}\right| \sim \frac{\varphi^{2n+2}}{5}\times\frac{\sqrt{5}}{\varphi^{2n+2}} \sim \frac{1}{\sqrt{5}}$$

$U_{n+1} = \cos U_n$, $U_1 \in [-1,1]$, $U_2 \in [0,1]$

Soit $\ell \in [0,1]$: $\cos \ell = \ell$ (TVI)

Point fixe attractif :

$$|U_{n+1}-\ell| = |\cos U_n - \cos \ell| \underset{IAF}{\leq} K\,|U_n - \ell| \qquad K = \sup_{t\in[0,1]} |\sin t| < 1$$

donc $\forall n \in \mathbb{N}$, $|U_{n+2}-\ell| \leq K^n |U_2-\ell| \to 0$

<u>Données</u> $f \in \mathcal{C}^1(I,I)$, $\ell \in I$

$|f'(\ell)| = 1$ neutre ?

$|f'(\ell)| < 1$ ℓ est attractif

On va mq :

① $\exists\, \delta > 0$, $\forall U_0 \in [\ell-\delta, \ell+\delta]$

$(U_n)\begin{cases} U_0 \\ U_{n+1} = f(U_n)\end{cases}$ converge vers ℓ.

Soit K tq. $|f'(\ell)| < K < 1$

Par $\mathcal{C}^0$ de f', il existe $\delta > 0$

t.q. $\forall t \in [\ell-\delta, \ell+\delta]$, $|f'(t)| \leq K$

De là, si $x \in [\ell-\delta, \ell+\delta]$,

$$|f(x)-\ell| = |f(x)-f(\ell)| \underset{IAF}{\leq} K\,|x-\ell|$$

$U_1 \in [\ell-\varepsilon, \ell+\varepsilon]$ stable

Il vient : $|U_1-\ell| = |f(U_0)-f(\ell)| \leq K\,|U_0-\ell|$

$|U_2-\ell| \leq K^2 |U_0-\ell|$

$\vdots$

$|U_n-\ell| \leq K^n |U_0-\ell|$ $\quad (U_n) \to \ell$

Points répulsifs : $|f'(\ell)| > 1$

Alors $(u_n) \to \ell \iff (u_n)$ stationnaire à ℓ

($\Leftarrow$) clair car $f(\ell) = \ell$.

($\Rightarrow$) ABS : Si $\exists n$, $u_n = \ell$, alors $\forall m > n$,
$u_m = \underbrace{\ell}_{\text{fixe}}$ NON !

Si $\forall n$, $u_n \neq \ell$, on regarde :

$$\left|\frac{u_{n+1}-\ell}{u_n-\ell}\right| = \left|\frac{f(u_n)-f(\ell)}{u_n-\ell}\right| \to |f'(\ell)| > 1$$

Soit $K = \frac{1+|f'(\ell)|}{2} > 1$

$$\exists N,\ \forall n \geq N,\ \left|\frac{u_{n+1}-\ell}{u_n-\ell}\right| \geq K$$

$$\forall n \geq N,\ |u_{n+1}-\ell| \geq K|u_n-\ell|$$

$$|u_{N+p}-\ell| \geq K^p|u_N-\ell| \to +\infty$$

NON !